Solving complex-valued ℓ_0 minimization problems with constant modulus constraints

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Joint work with
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Introduction

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We want to solve the following problem

Constant Modulus Program (CMP)

$$\min_{\mathbf{x}\in\mathbb{C}^N} \|\mathbf{x}\|_0$$
$$\text{s. t. } \|\mathbf{b}-\mathbf{A}\mathbf{x}\|_2 \leq \sqrt{\delta},$$
$$|x_n| \in \{0, c\}, \quad \forall n \in [N],$$

with $\mathbf{A} \in \mathbb{C}^{K\times N}$, $\mathbf{b} \in \mathbb{C}^K$, $\delta \in \mathbb{R}$, $c \in \mathbb{R}$, $[N] = \{1, \dots, n\}$ and $\|\mathbf{x}\|_0$ denoting the number of nonzero entries in $\mathbf{x}$.

- In the problem we could also allow additional constraints, such as linear (in-)equalities.
- Throughout this talk, we assume w.l.o.g. that $c = 1$.

Discrete Optimization

Motivation

- Generalization of compressed sensing: search sparse solutions of an optimization problem $\Rightarrow$ use ℓ_0-term in objective function.
- Note that the constant modulus constraint is nonconvex.
- We want to solve this problem exactly.
- constant modulus often appears in signal processing (e.g., phase retrieval).

MINLP–Reformulation

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- auxiliary binary variables $\mathbf{y} = [y_1, y_2, \dots, y_N]^\mathsf{T} \in \{0,1\}^N$.
- $w_n = \mathrm{Re}[x_n]$ and $z_n = \mathrm{Im}[x_n]$ as real and imaginary part of x_n, respectively.
- $\mathbf{w} = [w_1, w_2, \dots, w_N]^\mathsf{T}$ and $\mathbf{z} = [z_1, z_2, \dots, z_N]^\mathsf{T}$.
- $\mathbf{A}^T = [\mathbf{a}_1, \dots, \mathbf{a}_K] \in \mathbb{C}^{N \times K}$.

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$$
\begin{aligned}
\min_{\mathbf{w},\mathbf{z}\in\mathbb{R}^N,\mathbf{y}} \sum_{n=1}^{N} y_n \quad \text{s.t.} \quad & \sum_{k=1}^{K} \Big(\mathrm{Re}[b_k] - \big(\mathrm{Re}[\mathbf{a}_k]^T\mathbf{w} - \mathrm{Im}[\mathbf{a}_k]^T\mathbf{z}\big) \Big)^2 \\
& + \Big(\mathrm{Im}[b_k] - \big(\mathrm{Re}[\mathbf{a}_k]^T\mathbf{z} + \mathrm{Im}[\mathbf{a}_k]^T\mathbf{w}\big) \Big)^2 \leq \delta, \\
& w_n^2 + z_n^2 \leq y_n, \quad \forall\, n \in [N], \\
& w_n^2 + z_n^2 \geq y_n, \quad \forall\, n \in [N], \\
& y_n \in \{0,1\}, \quad \forall\, n \in [N].
\end{aligned}
$$

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Solving the resulting MINLP

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- Apply spatial branch-and-cut framework.
- Error bound (ℓ_2-norm) constraint is convex, even in the complex case, as it can be recast as a second-order cone constraint.
  - second-order cone: $C_k = \left\{ [x_1, \dots, x_k, x_{k+1}]^T \in \mathbb{R}^{k+1} \;:\; \|[x_1, \dots, x_k]^T\|_2 \leq x_{k+1} \right\}$.
  - second-order cone constraint: $\|A_i x + b_i\|_2 \leq c_i^T x + d_i$, $A_i \in \mathbb{R}^{(n_i-1)\times n}, b_i \in \mathbb{R}^{n_i-1}, c_i \in \mathbb{R}^n, d_i \in \mathbb{R}$.
- Upper bound parts of modulus constraints are convex quadratic constraints.
- Lower bound parts of modulus constraints are nonconvex quadratic constraints.

Solving the resulting MINLP

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- Upper bound parts of modulus constraints are convex quadratic constraints.
- Lower bound parts of modulus constraints are nonconvex quadratic constraints.

$\Rightarrow$ exploit structure of modulus constraints to develop specialized branching and propagation (and some very easy separation) to enforce modulus constraints.

The LP relaxation

Another problem: poor linear relaxation.

$\Rightarrow$ Use additional linear inequalities that strengthen the LP relaxation.

The LP relaxation

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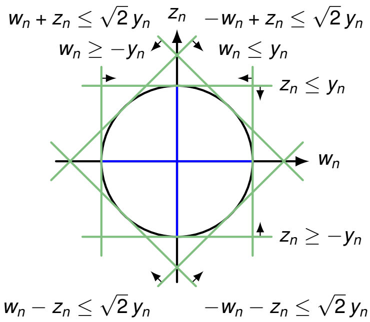

$$
\begin{aligned}
-y_n \leq w_n &\leq y_n, \\
-y_n \leq z_n &\leq y_n, \\
w_n + z_n &\leq \sqrt{2}\, y_n, \\
w_n - z_n &\leq \sqrt{2}\, y_n, \\
-w_n + z_n &\leq \sqrt{2}\, y_n, \\
-w_n - z_n &\leq \sqrt{2}\, y_n.
\end{aligned}
$$

Enforcing Modulus Constraints

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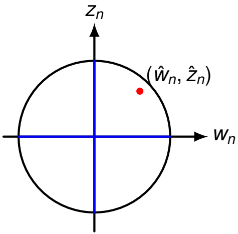
If the solution $(\hat{\mathbf{w}}, \hat{\mathbf{z}}, \hat{\mathbf{y}})$ of the LP relaxation violates $w_n^2 + z_n^2 \geq y_n$ for some $n \in [N]$:

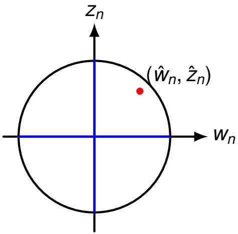


Discrete Optimization

Enforcing Modulus Constraints

If the solution $(\hat{\mathbf{w}}, \hat{\mathbf{z}}, \hat{\mathbf{y}})$ of the LP relaxation violates $w_n^2 + z_n^2 \geq y_n$ for some $n \in [N]$:

- If binary variable $\hat{y}_n$ fixed to zero: set $\hat{w}_n, \hat{z}_n$ to zero as well.



Enforcing Modulus Constraints

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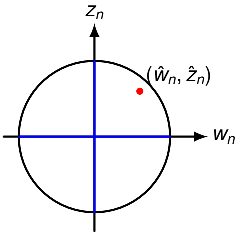
If the solution $(\hat{\mathbf{w}}, \hat{\mathbf{z}}, \hat{\mathbf{y}})$ of the LP relaxation violates $w_n^2 + z_n^2 \geq y_n$ for some $n \in [N]$:

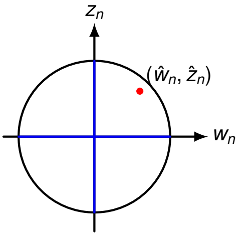
- If binary variable $\hat{y}_n$ fixed to zero: set $\hat{w}_n, \hat{z}_n$ to zero as well.
- If bounds of the continuous variables w_n and z_n are not yet restricted to one of the four orthants: create four branching nodes, one for each orthant.



Enforcing Modulus Constraints

If the solution $(\hat{\mathbf{w}}, \hat{\mathbf{z}}, \hat{\mathbf{y}})$ of the LP relaxation violates $w_n^2 + z_n^2 \geq y_n$ for some $n \in [N]$:

- If binary variable $\hat{y}_n$ fixed to zero: set $\hat{w}_n, \hat{z}_n$ to zero as well.
- If bounds of the continuous variables w_n and z_n are not yet restricted to one of the four orthants: create four branching nodes, one for each orthant.
- Otherwise:
  1. Propagation,
  2. Separation,
  3. Branching.

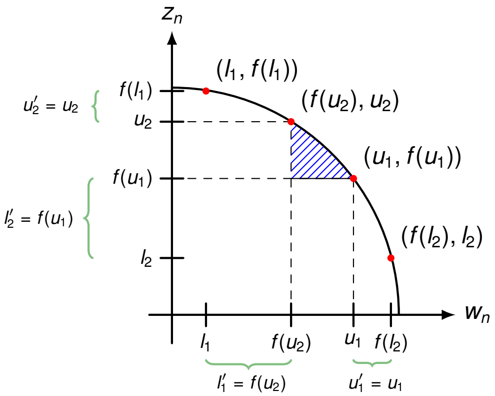


Propagation

If y_n is fixed to one:
l_1, u_1, l_2, u_2 current lower and upper bounds of w_n and z_n.

$$l_1' = \max\{l_1, f(u_2)\},$$
$$u_1' = \min\{u_1, f(l_2)\},$$
$$l_2' = \max\{l_2, f(u_1)\},$$
$$u_2' = \min\{u_2, f(l_1)\},$$

where $f(x) = \sqrt{1 - x^2}$.
$w_n^2 + z_n^2 \geq y_n$ needs to be fulfilled.



Propagation

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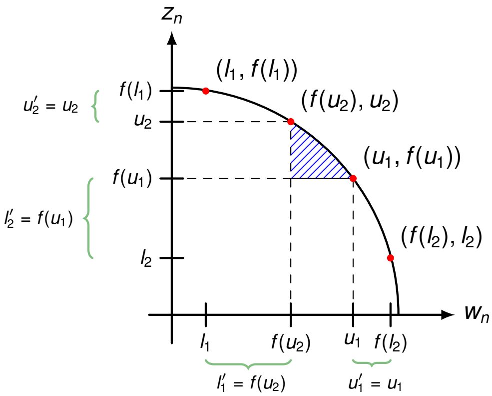
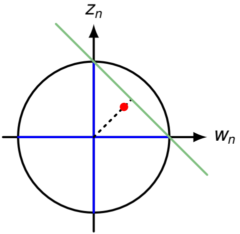


⇒ Use l_1', u_1', l_2' and u_2' as new lower and upper bounds of w_n and z_n.

Separation and Branching

If $\hat{w}_n + \hat{z}_n < \hat{y}_n$, add the cut $w_n + z_n \geq y_n$ to the LP relaxation.

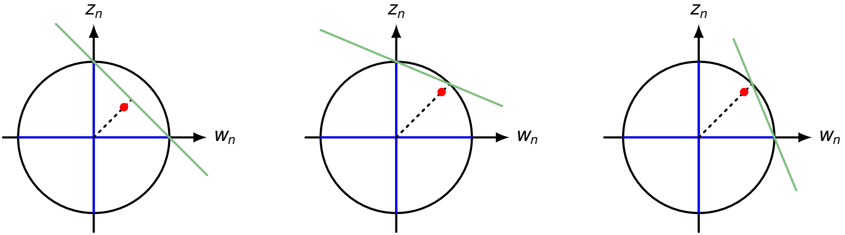


Separation and Branching

If $\hat{w}_n + \hat{z}_n < \hat{y}_n$, add the cut $w_n + z_n \geq y_n$ to the LP relaxation.
If $\hat{w}_n + \hat{z}_n \geq \hat{y}_n$, create two branching nodes with the inequalities

$$-\frac{\hat{z}_n{}' - 1}{\hat{w}_n{}'} \cdot w_n + z_n = 1, \qquad \frac{\hat{z}_n{}'}{\hat{w}_n{}' - 1} \cdot w_n - z_n = \frac{\hat{z}_n{}'}{\hat{w}_n{}' - 1},$$

where $(\hat{w}_n, \hat{z}_n)$ is the current LP solution, and $\hat{w}_n{}' = \frac{\hat{w}_n}{\sqrt{\hat{w}_n{}^2 + \hat{z}_n{}^2}}$, $\hat{z}_n{}' = \frac{\hat{z}_n}{\sqrt{\hat{w}_n{}^2 + \hat{z}_n{}^2}}$.



Specialized Approach – Some remarks

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- Enforcement of binary variables, second-order cone (SOC) constraint and the (convex) upper bound part of the quadratic modulus constraints is prioritized over the (nonconvex) lower bound part of the modulus constraints.
- A modulus constraint for enforcing is selected by a "most infeasible" rule: Enforce a modulus constraint $w_{\bar{n}}^2 + z_{\bar{n}}^2 \geq y_{\bar{n}}$, $\bar{n} \in [N]$, with largest violation

$$\rho(n) = \hat{y}_n - (\hat{w}_n^2 + \hat{z}_n^2).$$

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A heuristic

Recall the problem

$$\min \; \|\mathbf{x}\|_0 \;\text{ s. t. }\; \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 \leq \sqrt{\delta}, \quad |x_n| \in \{0, 1\}, \; \forall n \in [N].$$

Basic idea: fix sparsity level M, try iteratively to create an M-sparse solution by guessing and updating. If not successful, increase M by one and restart.

A heuristic

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Basic idea: fix sparsity level M, try iteratively to create an M-sparse solution by guessing and updating. If not successful, increase M by one and restart.

- guess solution: draw at random M indices and randomly choose their values on the unit circle, set the remaining values to zero.
- update solution: For two randomly chosen indices, one with non-zero value, one with zero value, recompute their values with a least squares problem where all but these two indices are fixed. Keep the smaller of the two new values, possibly swap the zero and non-zero index.

A heuristic

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Recall the problem

$$\min \|\mathbf{x}\|_0 \text{ s. t. } \|\mathbf{b} - \mathbf{Ax}\|_2 \leq \sqrt{\delta},$$
$$|x_n| \in \{0, 1\}, \forall n \in [N].$$

- guess 1000 starting solutions, update each 1000 times.
- If desired error bound $\sqrt{\delta}$ is reached: break.
- Use solution as initial primal bound in the branch-and-cut tree.

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Application example: Joint Antenna Selection and Phase-only Beamforming

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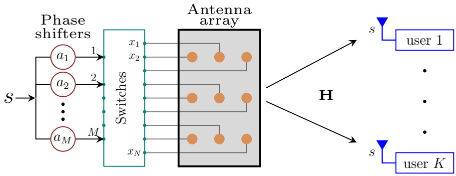
- Sensor array with N antenna elements, M phase shifters, $M \ll N$.
- $\mathbf{x} \in \mathbb{C}^N$ is the transmit signal vector, $x_n = |x_n| e^{i\psi_n}$.
- channel matrix $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_K] \in \mathbb{C}^{N \times K}$.
- Analog beamformers $\mathbf{a} = [a_1, a_2, \dots, a_M]^T \in \mathbb{C}^M$, assume $|a_m| = c$ for $m \in [M]$.
- if nth antenna element connected to mth phase shifter: $x_n = a_m$, else: $x_n = 0$.



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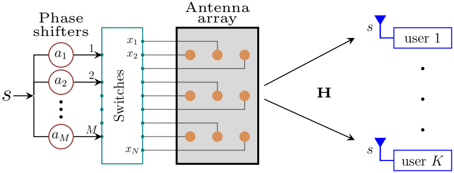
- if nth antenna element connected to mth phase shifter: $x_n = a_m$, else: $x_n = 0$.

- K single antenna users that need to be served.
- desired output at users: $\mathbf{s} = [s_1, s_2, \dots, s_K]^\mathrm{T}$.
- actual output at users: $\hat{\mathbf{s}} = \mathbf{H}^\mathrm{T}\mathbf{x} + \mathbf{n}$.



Application example: Joint Antenna Selection and Phase-only Beamforming

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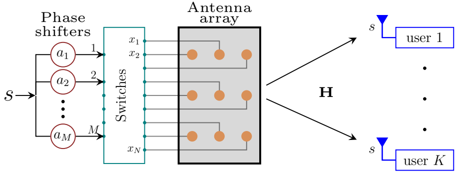
Our goal

- Design the optimal phase values of the ABF **a**.
- Assign appropriate antenna elements to the phase shifters to serve users in the network.
- Minimize the number of active antennas.
- Keep the root-mean-square error between the desired and actual output at the user below $\sqrt{\delta}$.



Mathematical model

$$\min_{\mathbf{x}\in\mathbb{C}^N} \|\mathbf{x}\|_0$$

$$\text{s. t. } \left\|\mathbf{s} - \mathbf{H}^T\mathbf{x}\right\|_2 \leq \sqrt{\delta},$$

$$|x_n| \in \{0, c\}, \quad \forall n \in [N].$$

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We have implemented the following parts:

- A reader that sets up the problem. It builds the error bound constraint as
  - either general quadratic constraint,
  - or as SOC constraint: auxiliary variables that represent $\mathbf{Ax} - \mathbf{b}$ are needed.
  - SOC-representation in SCIP: $\sqrt{\gamma + \sum_{i=1}^{n} (\alpha_i (x_i + \beta_i))^2} \leq \alpha_{n+1} (x_{n+1} + \beta_{n+1})$
- A probdata structure to save **A**, **b** and δ.
- A constraint handler for the branching, propagation and the separation methods for constant modulus constraints.
- The constraint handler has a propagation and enforcement method, the branching and separation is done in the enforcement method.
- A heuristic plugin.

Some computational results

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First testset: 48 instances, $N \in \{16, 32, 48, 64\}$, $K \in \{2, 3, 4\}$, $\delta^2 \in \{0.1, 0.2\}$
Second testset: 81 instances, $N \in \{32, 48, 64\}$, $K \in \{4, 5, 6\}$, $\delta^2 \in \{0.1, 0.2, 0.3\}$

First testset:

	SCIP 4.0.1			SCIP 5.0.1		
Setting	#opt	#nodes	time (s)	#opt	#nodes	time (s)
default	48	3309	36.4	46	2787	43.1
default + SOC	48	2575	26.0	48	2478	29.1
default + heur	48	396	21.5	47	448	21.9
default + SOC + heur	48	323	21.1	48	364	18.8
mod handling	48	3283	28.3	47	3398	35.5
mod handling + SOC	48	2762	18.4	48	2313	19.7
mod handling + heur	48	427	19.3	48	507	18.0
mod handling + SOC + heur	48	317	17.9	48	361	15.2

Second testset:

	SCIP 4.0.1			SCIP 5.0.1		
Setting	#opt	#nodes	time (s)	#opt	#nodes	time (s)
default	47	126710	1058.8	35	38884	1232.2
default + SOC	47	149482	812.0	40	60565	984.2
default + heur	56	29122	444.4	45	15310	590.2
default + SOC + heur	54	33296	455.9	51	18000	472.0
mod handling	43	154338	1070.4	34	85619	1257.9
mod handling + SOC	55	167485	666.0	49	138182	746.2
mod handling + heur	55	35912	433.6	54	28203	461.2
mod handling + SOC + heur	59	40334	360.2	58	36938	306.9

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Summary and further research

Conclusion:

- Specialized methods for handling modulus constraints result in speed-up.
- A good primal bound has a (very) high impact on the solution process, as significantly less nodes are needed.
- Explicitly building the convex error bound term as SOC greatly fastens the solution process.

Summary and further research

Conclusion:

- Specialized methods for handling modulus constraints result in speed-up.
- A good primal bound has a (very) high impact on the solution process, as significantly less nodes are needed.
- Explicitly building the convex error bound term as SOC greatly fastens the solution process.

Outlook:

- Let the heuristic also run in deeper nodes of the branch-and-cut tree and initialize it with current LP solution.
- Closely connected: vary the two parameters for the number of guesses and updates.
- Apply this framework to phase retrieval: For a measured modulus $|\mathbf{x}|$ of a complex signal $\mathbf{x} = |\mathbf{x}|e^{i\psi}$ find the phase ψ satisfying a set of constraints.

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Thanks for your attention!

Discrete Optimization

References

[WSA'18] T. Fischer, G. Hegde, F. Matter, M. Pesavento, M. E. Pfetsch, A. M. Tillmann, "Joint Antenna Selection and Phase-only Beamforming using Mixed-Integer Nonlinear Programming", in *22nd International ITG Workshop on Smart Antennas (WSA)*, Bochum, Germany, March 2018. [accepted]