Brachistochrone

Introduction

The *Brachistochrone* problem asks for the curve that a particle needs to follow, only under the presence of gravity, to go from point A to B in the least amount of time. We will solve this problem with the help of SCIP.

Problem formulation

Assume $A = (0, 1)$, $B = (1, 0)$. Given a curve $y(x)$ such that $y(0) = 1, y(1) = 0$, we can assign to it the time that a particle dropped at A takes to get to B following $y(x)$. Let us call this time T (though T_y would be better).
To find out T, notice that time is distance covered divided by the velocity. Let us find the distance covered first. For that write the curve as $x \to (x, y(x))$. The length covered from $x = 0$ to $x = \bar{x}$ is $s(\bar{x}) = \int_0^{\bar{x}} \sqrt{1 + {y'}^2(\alpha)} d\alpha$.
Now, $v(t) = \frac{ds}{dt}$ and so

$$T = \int_0^T dt$$

Using the change of variable (abussing notation) $t = t(s)$ and noting that $t = 0 \Leftrightarrow s = 0$ and $t = T \Leftrightarrow s = s(1)$, we get

$$T = \int_0^T dt = \int_0^{s(1)} \frac{dt}{ds} ds = \int_0^{s(1)} \frac{1}{v(s)} ds$$

Finally, using the change of variables $s = s(x)$, we obtain

$$T = \int_0^T dt = \int_0^{s(1)} \frac{1}{v(s)} ds = \int_0^1 \frac{\sqrt{1 + {y'}^2(x)}}{v(x)} dx$$

To find v we can use that the energy of the system is constant. The energy is the sum of the kinetic ($\frac{1}{2}mv^2$) and potential energy (mgy). At the beginning, the energy is $E(0) = \frac{1}{2}mv(0)^2 + mgy(0) = mg$. Then at x, $E(x) = \frac{1}{2}mv(x)^2 + mgy(x)$ and since it is constant, $E(0) = E(x)$, which implies $v = \sqrt{2g(1-y)}$
Then, given a curve y, we obtain

$$T = \int_0^1 \frac{\sqrt{1 + {y'}^2}}{\sqrt{2g(1-y)}} dx$$

and our task is to find the curve y that minimizes T,

$$\min_y \int_0^1 \frac{\sqrt{1 + {y'}^2}}{\sqrt{2g(1-y)}} dx$$

With techniques from calculus of variations this problem can be solved analytically. The optimal curve, for this particular case, is given parametrically on θ by

$$\begin{aligned} x(\theta) &= 0.573(\theta - \sin(\theta)) \\ y(\theta) &= -0.573(1 - \cos(\theta)) \\ \theta &\in [0, 2.412] \end{aligned}$$

and the total time is $T = 0.583$

Discretization

To use SCIP we need to discretize the problem. For this, we divide the integral in small pieces:

$$\int_0^1 \frac{\sqrt{1 + {y'}^2}}{\sqrt{2g(1-y)}} dx = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \frac{\sqrt{1 + {y'}^2}}{\sqrt{2g(1-y)}} dx$$

Approximating $y(x)$ by a linear for $x \in [x_i, x_{i+1}]$, we can write it as $y(x) = m_i(x - x_i) + y_i$, where $m_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$. With this, one can show that

$$\int_0^1 \frac{\sqrt{1 + {y'}^2}}{\sqrt{2g(1-y)}} dx \approx \sqrt{\frac{2}{g}} \sum_{i=0}^{N-1} \frac{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}}{\sqrt{1 - y_{i+1}} + \sqrt{1 - y_i}}$$

Using SCIP

Your task is to write a model to solve the discretized problem with PySCIPOpt.
(Optional) Show how to obtain the discretization.